

# On Order-Preserving and Verbal Embeddings of the Group $\mathbb{Q}$

Arman Darbinyan    Vahagn H. Mikaelian

## Abstract

We show that there is an order-preserving embedding of the additive group of rational numbers  $\mathbb{Q}$  into a 2-generator group  $G$ . The group  $G$  can be chosen to be a solvable group  $G$  of length 3, which is a minimal result in the sense that it cannot be chosen to be neither solvable of length 2, nor a nilpotent group. For any non-trivial word set  $V \subseteq F_\infty$  there is an order-preserving verbal embedding of  $\mathbb{Q}$  into a 2-generator group  $G$ . The embeddings constructed are subnormal.

## 1 Introduction

The aim of this note is to add some additional properties to the explicit embedding of the additive group of rational numbers  $\mathbb{Q}$  into a 2-generator group  $G$  we constructed in [13]. Existence of such an explicit embedding of  $\mathbb{Q}$  was asked by de la Harpe and Bridson in Problem 14.10 (b) of the Kourovka Notebook [5], and the construction of [13] was to give a positive answer to this question.

It is natural to ask which properties of  $\mathbb{Q}$  can be “inherited” by  $G$ , and which additional options can the embedding have. One of most natural properties characterizing  $\mathbb{Q}$  is the linear order of rational numbers and, thus, it is reasonable to ask if the group  $G$  can be ordered in such a way that its order continues the natural order of rational numbers in the isomorphic image of  $\mathbb{Q}$  in  $G$ . Clearly, here we are interested in such a linear order on  $G$  which is adjusted with the multiplication of the group  $G$ , that is, is a full order relation in the sense of [17, 6, 7] (see definitions below).

The next option we add to the embedding is *verbality*. For the non-trivial word set  $V \subseteq F_\infty$  the embedding of the group  $H$  into the group  $G$  is said to

---

**Keywords.** Fully ordered group, verbal embedding of group, variety of groups, free nilpotent group, wreath products.

**2010 Math. Subject Classification.** 20D35, 20E05, 20E10, 20E22, 20F18, 20F60.

be  $V$ -verbal if the isomorphic image of  $H$  under this embedding lies in the verbal subgroup  $V(G)$  (see definition below). Thus, the concept of  $V$ -verbal embedding, suggested by Heineken [2, 4, 3], is the wide generalization of the notion of embedding into the commutator subgroup, embedding into the member of the lower central series, embedding of the  $n$ 'th derived subgroup, etc.. See also recent research on verbal embeddings in [8]-[16].

Another property of the embedding we deal with is how “economical” it can be in the following sense. The 2-generator group  $G$ , into which  $\mathbb{Q}$  is embedded, is a solvable group of length 3. We show that  $G$  cannot be replaced neither by a finitely generated nilpotent group of any class, nor by a finitely generated solvable group of length 2 (that is, by any finitely generated metabelian group). The latter fact continues Neumann’s example of Lemma 5.3 in [18]: the quasi-cyclic group  $Z(p^\infty)$  is an example of an infinitely generated abelian group, which cannot be embedded into a finitely generated metabelian group.

Since the notions used in this paper not always have standard definitions or notations in the literature, we bring here a brief list of definitions and references to the sources, where the detailed information and main properties can be found.

The group  $G$  is *fully ordered*, if a linear order relation  $<$  is defined on  $G$ , such that for arbitrary elements  $g_1, g_2 \in G$ ,  $g_1 < g_2$  implies  $g_1x < g_2x$  and  $xg_1 < xg_2$  for any  $x \in G$ . In the literature these days it is familiar to call the fully ordered groups “linearly ordered groups”, but we use the older terminology because here we use a few other linear order relations also, and some difference in terminology is useful. An embedding  $\varphi$  of the fully ordered group  $H$  into the fully ordered group  $G$  is said to be order-preserving, if for any  $h_1, h_2 \in H$  the relation  $h_1 < h_2$  holds in  $H$  if and only if  $\varphi(h_1) < \varphi(h_2)$  holds in  $G$ . For more information about fully ordered groups see [17, 6, 7].

*Cartesian* wreath product  $A \text{ Wr } B$  of two groups  $A$  and  $B$  is the semidirect product  $B \ltimes A^B$ , where  $A^B$  is the set of maps from  $B$  to  $A$ , and  $B$  acts on  $A^B$  by the rule: for arbitrary  $f : B \rightarrow A$ ,  $b \in B$ ;  $f^b(b_0) = f(b_0b^{-1})$  for any  $b_0 \in B$ . For more information about wreath products we refer to [19].

For a group  $G$  and for a word set  $V \subseteq F_\infty$  the  $V$ -verbal subgroup  $V(G)$  of  $G$  is the subgroup generated by all substitutions  $v(g_1, \dots, g_n) \in G$  for all words  $v(x_1, \dots, x_n) \in V$  and for all elements  $g_1, \dots, g_n \in G$ . When it is clear from context which word set  $V$  is assumed, we call the  $V$ -verbal subgroup a verbal subgroup. The variety corresponding to  $V$  is the variety generated by the factor group  $F_\infty/V(F_\infty)$  or, in other words, the set of all groups for which any  $v(x_1, \dots, x_n) \in V$  is an identity. A word set is said to be non-trivial if  $V(F_\infty) \neq \{1\}$ . For more information about word sets, varieties, identities see [19].

An embedding  $\varphi$  of the group  $H$  into the group  $G$  is said to be subnormal,

if  $\varphi(H)$  is a subnormal subgroup in  $G$ , that is, if there is a finite series of subgroups  $\varphi(H) = G_0 \leq G_1 \leq \dots \leq G_n = G$ , such that  $G_{i-1}$  is a normal subgroup in  $G_i$  for any  $i = 1, \dots, n$ .

## 2 The order-preserving explicit embedding construction

The following is the main theorem of the current paper:

**THEOREM 1.** *There is an order-preserving subnormal embedding of the additive group of rational numbers  $\mathbb{Q}$  into a fully ordered 2-generator group  $G$ . The group  $G$  can be chosen to be a solvable group  $G$  of length 3.*

Before giving the description of the embedding of  $\mathbb{Q}$  with the mentioned properties, let us state a lemma which we will use later on.

**LEMMA 1.** *Let  $A$  and  $B$  be fully ordered groups and  $X \leq A \text{ Wr } B$ . If for each  $b\varphi \in X$ ,  $\text{supp}(\varphi)$  is well-ordered, then  $X$  can be fully ordered.*

*Proof.* If  $b_1\varphi_1$  and  $b_2\varphi_2$  belong to  $X$ , then  $b_1\varphi_1 < b_2\varphi_2$  is defined as:  $b_1 < b_2$  or  $b_1 = b_2$  and  $\phi_1(b) < \phi_2(b)$  for the least  $b$  for which  $\phi_1(b) \neq \phi_2(b)$ .

Let  $U_1$  be the subset of the union  $\text{supp}(\phi_1) \cup \text{supp}(\phi_2)$  consisting of elements  $b$  for which  $\phi_1(b) < \phi_2(b)$ . Since  $\text{supp}(\phi_1)$  and  $\text{supp}(\phi_2)$  are well-ordered,  $U_1$  has a least element  $u_1$ . Similarly define  $U_2$  of elements  $b$  for which  $\phi_2(b) < \phi_1(b)$  and take its least element  $u_2$ . Since  $B$  is fully ordered, its order is linear and either  $u_1 < u_2$ , in which case  $\phi_1 < \phi_2$ , or  $u_2 < u_1$ , in which case  $\phi_2 < \phi_1$ . The only exception is the case when  $\phi_1 = \phi_2$ , of course. Thus, the order  $<$  is linear on  $X$ .

We must show that this order is also full order. Let us take an arbitrary element  $b_3\varphi_3 \in X$  and show that if  $b_1\varphi_1 < b_2\varphi_2$ , then  $b_1b_3\varphi_1^{b_3}\varphi_3 < b_2b_3\varphi_2^{b_3}\varphi_3$  and  $b_3b_1\varphi_3^{b_1}\varphi_1 < b_3b_2\varphi_3^{b_2}\varphi_2$ .

If  $b_1 < b_2$  then this is obvious. So we need to consider the case when  $b_1 = b_2$ ,  $\varphi_1 < \varphi_2$ . Hence it is sufficient to show that

$$\varphi_1^{b_3}\varphi_3 < \varphi_2^{b_3}\varphi_3 \quad (*)$$

and

$$\varphi_3^{b_1}\varphi_1 < \varphi_3^{b_1}\varphi_2. \quad (**)$$

We have:

$$\begin{aligned}
& \min\{b \in B \mid \varphi_1^{b_3}(b)\varphi_3(b) < \varphi_2^{b_3}(b)\varphi_3(b)\} \\
&= \min\{b \in B \mid \varphi_1(bb_3^{-1}) < \varphi_2(bb_3^{-1})\} \\
&= \min\{b \in B \mid \varphi_1(b) < \varphi_2(b)\}b_3 \\
&< \min\{b \in B \mid \varphi_1(b) > \varphi_2(b)\}b_3 \\
&= \min\{b \in B \mid \varphi_1^{b_3}(b)\varphi_3(b) \\
&> \varphi_2^{b_3}(b)\varphi_3(b),
\end{aligned}$$

hence (\*) is proved. And also:

$$\begin{aligned}
& \min\{b \in B \mid \varphi_3(b)\varphi_1(b) < \varphi_3^{bb_1^{-1}}(b)\varphi_2(b)\} \\
&= \min\{b \in B \mid \varphi_1(b) < \varphi_2(b)\} \\
&> \min\{b \in B \mid \varphi_1(b) > \varphi_2(b)\} \\
&= \min\{b \in B \mid \varphi_3(b)\varphi_1(b) \\
&> \varphi_3^{bb_1^{-1}}(b)\varphi_2(b),
\end{aligned}$$

hence (\*\*) is also proved. So Lemma 1 is proved.  $\square$

*Remark 1.* In the proof of the Lemma 1 we not only proved that the subgroup  $X$  can be fully ordered, but also suggested an explicit group order. Also, if the intersection of  $X$  and the first copy of  $A$  in  $AWrB$  contains  $a_1, a_2 \in A$ , then  $a_1 < a_2$  in  $A \Rightarrow a_1 < a_2$  in  $X$ .

*Remark 2.* The requirement of being well-ordered of support in the statement of Lemma 1 is necessary. Moreover, it is true that the wreath product of two infinite groups can not be fully ordered [17].

Bellow we have constructed an embedding of  $\mathbb{Q}$  into a 2-generator subgroup of  $W = (\mathbb{Q} \text{ Wr } C) \text{ Wr } Z$ , where  $C = \langle c \rangle$  is an infinite cyclic group generated by  $c$ , and  $Z = \langle z \rangle$  is an infinite cyclic group generated by  $z$ .

Firstly, for each positive integer  $n$ , choose in the base subgroup  $\mathbb{Q}^C$  of the Cartesian wreath product  $\mathbb{Q} \text{ Wr } C$  the elements  $\varphi_n$  and  $\tau_n$ :

$$\varphi_n(c^i) = \begin{cases} \frac{1}{n} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases} \quad \tau_n(c^i) = \begin{cases} 0 & \text{if } i < 0, \\ \frac{-1}{n} & \text{if } i \geq 0. \end{cases}$$

The reason of such selection is in the following relations:

$$[\tau_n, c] = \varphi_n, [\tau_m, \tau_n] = 1 \text{ for any } n, k > 0. \tag{1}$$

The first of the relations (1) trivially follows from

$$[\tau_n, c](c^i) = \tau_n^{-1} \tau_n^c(c^i) = \begin{cases} -0 + 0 = 0 & \text{if } i < 0, \\ \frac{1}{n} + 0 = \frac{1}{n} & \text{if } i = 0, \\ \frac{1}{n} + \frac{-1}{n} = 0 & \text{if } i > 0. \end{cases}$$

The second of the relations (1) trivially follows from the fact that  $\mathbb{Q}^C$  is abelian.

In the base subgroup  $(\mathbb{Q} \text{ Wr } C)^Z$  of  $W$ , take an element  $\alpha$  defined as

$$\alpha(z^j) = \begin{cases} 1 = 1_{\mathbb{Q} \text{ Wr } C} & \text{if } j < 0, \\ c & \text{if } j = 0, \\ \tau_j & \text{if } j > 0. \end{cases}$$

Put  $G = \langle \alpha, z \rangle$  and define the embedding  $\Phi : \mathbb{Q} \rightarrow G$  as

$$\Phi : \frac{m}{n} \mapsto [z^n \alpha z^{-n}, \alpha]^m = [\alpha^{z^{-n}}, \alpha]^m \text{ for any } \frac{m}{n} \in \mathbb{Q}, n > 0.$$

That  $\Phi$  is a homomorphism and an injection could be checked directly. But to avoid very long calculations, we consider the structure of the commutator  $[\alpha^{z^{-n}}, \alpha]$  first:

$$[\alpha^{z^{-n}}, \alpha](z^j) = [\alpha(z^{j+n}), \alpha(z^j)] = \begin{cases} [1, 1] = 1 & \text{if } j < -n, \\ [c, 1] = 1 & \text{if } j = -n, \\ [\tau_{j+n}, 1] = 1 & \text{if } -n < j < 0, \\ [\tau_n, c] = \varphi_n & \text{if } j = 0, \\ [\tau_{j+n}, \tau_j] = 1 & \text{if } j > 0. \end{cases}$$

This means that  $[\alpha^{z^{-n}}, \alpha]$  is nothing else but the image  $\phi_n^*$  of the coordinate element  $\varphi_n$  in the “the first copy” of the group  $\mathbb{Q} \text{ Wr } C$  in  $W$ :

$$\varphi_n^*(z^j) = \begin{cases} \varphi_n & \text{if } j = 0, \\ 1 = 1_{\mathbb{Q} \text{ Wr } C} & \text{if } j \neq 0. \end{cases}$$

Therefore the elements

$$\Phi(\mathbb{Q}) = \{ \Phi(\frac{m}{n}) = (\varphi_n^*)^m \mid \frac{m}{n} \in \mathbb{Q}, n > 0 \}$$

do form a subgroup isomorphic to  $\mathbb{Q}$  in  $G$ , and the mapping  $\Phi$  is injective.

Finally, it easily follows from the equalities

$$\Phi\left(\frac{1}{n}\right)\Phi\left(\frac{1}{n'}\right) = \varphi_n^* \varphi_{n'}^*, \quad \Phi\left(\frac{m}{n}\right) = (\varphi_n^*)^m = \underbrace{\varphi_n^* + \dots + \varphi_n^*}_{n \text{ times}}$$

that  $\Phi$  is a homomorphism.

Each element of  $G$  can be presented in the form

$$z^k (\alpha^{z^{k_1}})^{n^1} \cdot \dots \cdot (\alpha^{z^{k_s}})^{n^s}, \quad (2)$$

which follows from the fact that

$$\alpha^n z^m = z^m (\alpha^{z^m})^n.$$

Let us denote the product of  $\alpha$ -factors of (2) by  $\tilde{\alpha}$ .

$$\tilde{\alpha}(z^i) = 1_{\mathbb{Q} \operatorname{Wr} C}, \text{ if } i < \min\{k_1, \dots, k_s\}. \quad (3)$$

It is clear that  $G \leq T \operatorname{Wr} \mathbb{Z}$ , where  $T = \langle c, \tau_i \mid i \in \mathbb{Z}, i > 0 \rangle$  and  $\operatorname{supp}(\tilde{\alpha})$  is well-ordered.

Hence, by Lemma 1, in order to show that  $G$  is fully ordered with order relation defined in the proof of Lemma 1, we only need to show that  $T$  is fully ordered.

Similarly to 2 each element of  $T$  can be presented in the form:

$$c^{k_0} (\tau_{i_1}^{c^{k_1}})^{n_1} (\tau_{i_2}^{c^{k_2}})^{n_2} \cdot \dots \cdot (\tau_{i_m}^{c^{k_m}})^{n_m} \quad \text{where } k_1 \leq k_2 \leq \dots \leq k_m,$$

which follows from  $(\tau_i^{c^j})^n c^k = c^k (\tau_i^{c^{k+j}})^n$  and from the fact that  $(\tau_{i_p}^{c^{k_p}})^{n_p}$  commutes with  $(\tau_{i_q}^{c^{k_q}})^{n_q}$ , where  $1 \leq p, q \leq m$ . Obviously  $T$  is a subgroup of  $\mathbb{Q} \operatorname{Wr} C$ ,

If we denote the product of  $\tau$ -factors of (3) by  $\tilde{\beta}$ , then

$$\tilde{\beta}(c^i) = \begin{cases} 0 & \text{if } i < k_1, \\ \frac{n_1}{i_1} + \frac{n_2}{i_2} + \dots + \frac{n_s}{i_s} & \text{if } k_s > i \geq k_{s-1}. \end{cases} \quad (4)$$

Note that we not only proved that  $G$  is fully ordered, but also suggested an explicit order relation. Indeed, we firstly embedded  $\mathbb{Q}$  into a countably generated subgroup  $T$  of  $\mathbb{Q} \operatorname{Wr} C$ , then embedded  $T$  into a 2-generator subgroup  $G$  of  $(\mathbb{Q} \operatorname{Wr} C) \operatorname{Wr} Z$  and it follows from (3) and Lemma 1 that the first embedding preserves full order, and it follows from (4) and Lemma 1 that the second embedding also preserves full order.

The embedding  $\Phi$  of  $\mathbb{Q}$  into  $G$  described above is subnormal. As a subgroup of an abelian group  $\langle \phi_n \mid n \in \mathbb{N} \rangle$  is normal in the first copy of  $\mathbb{Q}$

in  $\mathbb{Q} \text{ Wr } C$ , the first copy is normal in  $\mathbb{Q}^C$ , and  $\mathbb{Q}^C$  is a normal subgroup of  $\mathbb{Q} \text{ Wr } C$ . Hence  $\mathbb{Q}^\Phi = \langle \phi_n^* \mid n \in \mathbb{N} \rangle$  is subnormal in the first copy of  $\mathbb{Q} \text{ Wr } C$  in  $(\mathbb{Q} \text{ Wr } C) \text{ Wr } Z$ . Therefore,  $\mathbb{Q}^\Phi$  is subnormal in  $G$ , because the first copy of  $\mathbb{Q} \text{ Wr } C$  in  $(\mathbb{Q} \text{ Wr } C) \text{ Wr } Z$  is subnormal subgroup of  $(\mathbb{Q} \text{ Wr } C) \text{ Wr } Z$ .

And  $G$  is a solvable group of length at most 3 since  $(\mathbb{Q} \text{ Wr } C) \text{ Wr } Z$  is a solvable group of length 3.

This concludes the proof of Theorem 1.

### 3 Additional properties of the embedding

Now we will examine additional properties of  $\mathbb{Q}$  and  $G$ .

*Property 1.*  $G$  is torsion free.

*Proof.* It follows from the fact that  $(\mathbb{Q} \text{ Wr } C) \text{ Wr } Z$  is torsion-free group.  $\square$

Product of two group varieties defined as a variety consisting of all extensions of a group from the first variety by a group of the second variety (see [19]). As we know wreath product  $A \text{ Wr } B$  of two groups  $A$  and  $B$  is an extension of the Cartesian product  $A^B$  by  $B$ . Hence, the following property is true.

*Property 2.*  $G$  belongs to the variety  $\mathfrak{AA}\mathfrak{A} = \mathfrak{S}_3$ , since  $\mathbb{Q}$ ,  $C$  and  $Z$  are abelian (by  $\mathfrak{S}_3$  we denoted the variety of solvable groups of length 3). This means that  $G$  is a solvable group of length 3, as we noted in proof of Theorem 1. The propositions below show that here  $\mathfrak{S}_3$  cannot be replaced by  $\mathfrak{S}_2$ .

The following propositions show that the embedding of  $\mathbb{Q}$  in  $G$  is the most economical, in the sense that  $\mathbb{Q}$  cannot be embedded into nilpotent group, or in a metabelian group.

**PROPOSITION 1.**  $\mathbb{Q}$  cannot be embedded into a finitely generated nilpotent group.

*Proof.* The fact follows from P.Hall's result [1], which states that every finitely generated nilpotent group satisfies the maximal condition for subgroups. Thus every subgroup of such a group is finitely generated. Since  $\mathbb{Q}$  is not finitely generated, it cannot be embedded into finitely generated nilpotent group.  $\square$

**PROPOSITION 2.**  $\mathbb{Q}$  cannot be embedded into a finitely generated metabelian group.

*Proof.* Suppose that the converse is true, that is to say  $\mathbb{Q}$  is embedded into a finitely generated metabelian group  $G$ . Evidently, a finitely generated abelian group satisfies the maximal condition for subgroups, also by the P.Hall's result [1], finitely generated metabelian groups satisfy the maximal condition for normal subgroups. Therefore the commutator  $G'$  is finitely generated and hence  $G' \cap \mathbb{Q}$  is finitely generated (therefore cyclic).

$G/G'$  is a finitely generated abelian group, therefore the subgroup  $\mathbb{Q}G'/G'$  of  $G/G'$  also is finitely generated. But we have  $\mathbb{Q}G'/G' \simeq \mathbb{Q}/G' \cap \mathbb{Q}$ , which is not finitely generated because  $\mathbb{Q}$  is not finitely generated. Contradiction.  $\square$

## 4 The verbal embedding of $\mathbb{Q}$

For definitions and basic facts about non-trivial word sets, verbal subgroups and embeddings see Introduction above and literature cited there.

**THEOREM 2.** *For any non-trivial set of words  $V$  there is an order-preserving subnormal embedding of the additive group of rational numbers  $\mathbb{Q}$  into a fully ordered 2-generator group  $G$ .*

For embedding construction purposes we need to find a fully ordered torsion free nilpotent group  $S$  with a non-trivial positive element  $a \in V(S)$ , as it is done in [9]. As a such group we take  $S = F_k(\mathfrak{N}_c)$ , where  $c$  is the least integer, such that  $\mathfrak{N}_c$  is not contained in the variety defined by  $V$  and  $k$  is such that  $S \notin \text{var}(F_\infty/V(F_\infty))$ . A full order relation can be defined in  $S$  (see [9], we omit the routine details to much shorten the proof since the exact method of construction of that embedding is immaterial for purposes of this proof).

We take an arbitrary non-trivial element  $a \in V(S) \neq 1$ . In any case we can assume  $a$  to be positive ( $1 < a$ ), for we always are in position to replace our order relation  $<$  by the inverse relation  $<^{-1}$  (for details see [9]).

As an element of  $V(S)$  our element  $a$  has the presentation

$$a = (v_1(a_{11}, \dots, a_{1t_1}))^{\varepsilon_1} \cdots (v_d(a_{d1}, \dots, a_{dt_d}))^{\varepsilon_d}$$

where  $\varepsilon_i = \pm 1, v_i \in V, a_{ij} \in S (i = 1, \dots, d; j = 1, \dots, t_i)$ .

Now let us consider the Cartesian wreath product  $\mathbb{Q} \text{ Wr } S$  and for each positive integer  $n$  define an element  $\chi_n$  and  $\psi_n$  as follows:

$$\psi_n(s) = \begin{cases} \frac{1}{n} & \text{if } s = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \chi_n(s) = \begin{cases} \frac{1}{n} & \text{if } s = a^i, i = 0, 1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the subgroup of  $\mathbb{Q} \text{ Wr } S$

$$T = \langle \chi_n, a_{ij} \mid n \in \mathbb{N}, i = 1, \dots, d; j = 1, \dots, t_i \rangle$$



LEMMA 2. *Let  $V$  be an arbitrary non-trivial word set and  $T = T(\mathbb{Q}, V)$  is that constructed above. Then:*

- (1)  $\mathbb{Q}$  can be embedded in  $T$  such that its image lies in  $V(T)$ .
- (2)  $T$  can be fully ordered, such that the order of  $\mathbb{Q}$  will be preserved by the embedding.

*Proof.* For the proof of the first part we just need to notice that  $\frac{m}{n} \mapsto \psi_n^m$  is an embedding of  $\mathbb{Q}$  into  $T$  and  $\psi_n = a^{-1}a^{\chi_n} \in V(T)$ , because  $a \in V(T)$  and  $V(T)$  is normal in  $T$ .

For the proof of the second part let us take an arbitrary element of  $T$

$$x_1 a_1 x_2 a_2 \dots x_k a_k \quad (5)$$

where  $k$  is some integer,  $x_i \in \langle \chi_n \mid n \in \mathbb{N} \rangle$   $i = 1, \dots, k$  and  $a_j \in \langle a_{pq} \mid p = 1, \dots, d; q = 1, \dots, t_p \rangle$ ,  $i, j = 1, \dots, k$ . Obviously each element of  $T$  can be presented in this form. Also it is obvious that  $\text{supp}(x_i), i = 1, \dots, k$  is well-ordered. Now let us transform the presentation (5) to the form:

$$a_1 a_2 \dots a_k x_1^{a_1 a_2 \dots a_k} x_2^{a_2 a_3 \dots a_k} \dots x_k \quad (6)$$

It is clear that  $\text{supp}(x_1^{a_1 a_2 \dots a_k} x_2^{a_2 a_3 \dots a_k} \dots x_k) \subseteq \bigcup_{i=1}^k \text{supp}(x_i^{a_i \dots a_k})$ . Now the proof follows from the Lemma 1, by taking into the account the fact that a finite union of well-ordered sets is well-ordered (thanks to the fact that the active group of the wreath product is linearly ordered, since it is fully ordered).  $\square$

For the later use let us note the following commutator identities:  $[\chi_i, \chi_j] = 1$ ,  $[a, \chi_i] = \psi_i$ .

The next step is to embed  $T$  into a subgroup of the Cartesian wreath product  $T \text{ Wr } C$ , where  $C = \langle c \rangle$  is an infinite cyclic group. Let us denote by  $\rho_g$  the element of the first copy of  $T$  in base group  $T^C$  corresponding to  $g \in T$ . In addition, define

$$\pi_g(c^i) = \begin{cases} g & \text{if } i \geq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then  $[\pi_{g^{-1}}, c] = \rho_g$  and it is important to note that the first copy of  $T$  lies in the derived subgroup of the group

$$D = \langle \pi_g, c \mid g \in T \rangle.$$

So we can embed  $T$  into  $D$  by the rule  $g \mapsto \rho_g$ , for all  $g \in T$ . Now we should

show that  $D$  can be fully ordered. Using the same transformation, that we have described above we can present every element of  $D$  in the form

$$d = c^k \pi_{g_1}^{c^{k_1}} \pi_{g_2}^{c^{k_2}} \cdot \dots \cdot \pi_{g_n}^{c^{k_n}}$$

It is easy to see that the support of the “right part” of  $D$  is well ordered, therefore by Lemma 1,  $D$  can be fully ordered. Now it can be checked directly, that the order described in the proof of Lemma 1 is preserved by the above described embedding.

Obviously  $D$  is countable. Let us enumerate the elements of  $D$ , such that

$$D = \{d_0, d_1, \dots, d_n, \dots; n \in \mathbb{N}\}.$$

Define an element  $\omega$  in  $D^Z$ :

$$\omega(z^i) = \begin{cases} d_k & \text{if } i = 2^k, k = 0, 1, 2, \dots, \\ 1 & \text{otherwise.} \end{cases}$$

For arbitrary  $d_n$  (that is, for every  $n$ )  $\omega^{(z^{-2^n})}(1) = d_n$  holds. So for each pair  $d_n$  and  $d_m$  we have

$$[\omega^{(z^{-2^n})}, \omega^{(z^{-2^m})}](1) = [d_n, d_m].$$

Furthermore, for arbitrary  $j \neq 0$ ,

$$[\omega^{(z^{-2^n})}, \omega^{(z^{-2^m})}](z^j) = 1$$

(see [9]). Thus every element of the derived subgroup  $D'$  belongs to the derived subgroup of a 2-generator group,

$$G = \langle \omega, z \rangle.$$

Now by using the same arguments as it was mentioned above, we can check that  $G$  can be fully ordered, such that the embedding of  $\mathbb{Q}$  into  $G$  preserves the “natural” order of  $\mathbb{Q}$ . Indeed, each element of  $G$  can be presented in the form

$$z^k (\omega^{z^{k_1}})^{n^1} \cdot \dots \cdot (\omega^{z^{k_s}})^{n^s}$$

and  $\left[ (\omega^{z^{k_1}})^{n^1} \cdot \dots \cdot (\omega^{z^{k_s}})^{n^s} \right] (z^i) = 1_T \text{ Wr } C$ , if  $i < \min\{k_1, \dots, k_s\}$ . Hence, supports of the “right parts” of the elements of  $G$  are well ordered, thus by Lemma 1  $G$  can be fully ordered.

Subnormality of the embedding of  $\mathbb{Q}$  described in this section can be shown analogously to the similar property of the embedding from the Section 2, again by taking into account the fact that the first copy of the passive group is subnormal in wreath product.

This concludes the proof of Theorem 2.

Observing the steps of the embedding construction, it is easy to note that the 2-generator group  $G$  belongs to the variety  $\mathfrak{AN}_c\mathfrak{A} \subseteq \mathfrak{S}_{c+3}$ , therefore  $G$  is a solvable group of length at most  $c + 3$ .

Arman Darbinyan  
 Department of Informatics and Applied Mathematics  
 Yerevan State University  
 Alex Manoogian 1  
 0025 Yerevan, Armenia  
 Email: arman.darbin@gmail.com

Vahagn H. Mikaelian  
 Department of Informatics and Applied Mathematics  
 Yerevan State University  
 Alex Manoogian 1  
 0025 Yerevan, Armenia  
 Email: v.mikaelian@gmail.com

## References

- [1] P. Hall, *Finiteness Conditions for Soluble Groups* . Proc. London Math. Soc., 419-436, 1954.
- [2] H. Heineken, *Normal embeddings of  $p$ -groups into  $p$ -groups*, Proc. Edinburgh Math. Soc. 35 (1992) , 309–314.
- [3] H. Heineken, *On normal embedding of subgroups.*, Geom. Dedicata 83, No.1-3, 211–216 (2000).
- [4] H. Heineken, V. H. Mikaelian, *On normal verbal embeddings of groups*, J. Math. Sci., New York, 100 (2000), 1, 1915–1924.
- [5] V. D. Mazurov, E.I.Khukhro, eds., *The Kourovka Notebook, Unsolved Problems in Group Theory*. 14th ed., Rossiiskaya Akademiya Nauk Sibirskoe Otdelenie, Institut Matematiki, Novosibirsk, 1999.
- [6] F. W. Levi, *Ordered groups*, Proc. Indian Acad. Sci., 16 (1942), 256–263.
- [7] F. W. Levi, *Contributions to the theory of ordered groups*, Proc. Indian Acad. Sci., 17 (1943), 199–201.
- [8] V. H. Mikaelian, *Subnormal embedding theorems for groups*, J. London Math. Soc., 62 (2000), 398–406.
- [9] V. Mikaelian, *On Embeddings of Countable Gnerealized Soluble Groups into Two-Generated Groups*. J. Algebra, 250 (2002), 1–17.
- [10] V. H. Mikaelian, *An embedding construction for ordered groups*, J. Austral Math. Soc. (A), 74 (2003), 379–392.
- [11] V. H. Mikaelian, *Infinitely many not locally soluble  $SI^*$ -groups*, Ricerche di Matematica, Univ. Studi Napoli, Naples, 52 (2003), 1–19.

- [12] V. H. Mikaelian, *On embedding properties of SD-groups*, International Journal of Mathematics and Mathematical Sciences, 2004:2 (2004) 65–76.
- [13] V. Mikaelian, *On a Problem on Explicit Embeddings of The Group  $\mathbb{Q}$* . International Journal of Mathematics and Mathematical Sciences, 2005, 2119–2123.
- [14] V. H. Mikaelian, *On finitely generated soluble non-Hopfian groups, an application to a problem of Neumann*, IJAC, International Journal of Algebra and Computations, 17 (2007), Nos. 5-6, 1107–1113.
- [15] V. H. Mikaelian, *On finitely generated soluble non-Hopfian groups*, Fundam. Prikl. Mat., 14:8 (2009), 185202 (in Russian and English).
- [16] V. H. Mikaelian, *Verbal embeddings and wreath products of groups*, Fundam. Prikl. Mat., accepted for publication (in Russian and English).
- [17] B. H. Neumann, *On Ordered Groups*. Amer.J.Math.71, 1-18, 1949.
- [18] B. H. Neumann, Hanna Neumann, *Embedding theorems for groups* J. London Math. Soc. 34 (1959), 465–479.
- [19] H. Neumann, *Varieties of Groups*. Springer-Verlag, 1967.
- [20] A. Yu. Ol'shanskii, *Efficient embeddings of countable groups*. Vestnik Moskov. Univ. Ser. I Mat. Mekh., 105, no. 2 (1989), 28–34. (Russian)
- [21] A. Yu. Ol'shanskii, *Embedding of countable periodic groups in simple 2-generator periodic groups*, Ukrain. Mat. Zh. 43 (1991), no. 7-8, 980–986 (Russian); translation in Ukrainian Math. J. 43 (1991), no. 7-8, 914–919 (1992).
- [22] D. J. S. Robinson, *A Course in the Theory of Groups*. Springer-Verlag, New York, 1996.